Optimal Sensing Precision in Ensemble and Unscented Kalman Filtering

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Introduction

We consider the problem of selecting an optimal set of sensor precisions to estimate the states of a non-linear dynamical system using an

- Ensemble Kalman filter
- Unscented Kalman filter

satisfying an upper bound on the state estimation error covariance.

- The sensor precision-selection problem for EnKF and UKF has not been addressed before.
- Formulated a convex optimization problem to determine the optimal sensor precision for a given upper bound on the state estimation error covariance.

System model

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Consider an input/output discrete-time stochastic system modeled by,

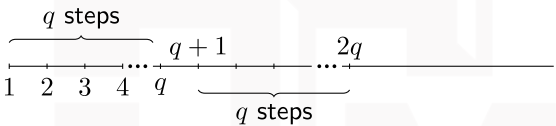
$$x_{k+1} = f_k(x_k, w_k), \quad (1a)$$

$$y_k = h_k(x_k) + v_k, \quad (1b)$$

- $w_k \in \mathbb{R}^{n_w}$ and $v_k \in \mathbb{R}^{n_y}$ are the process and measurement noise
- $\{w_k\}$ and $\{v_k\}$ are zero-mean, Gaussian, independent white random processes
- $w_k \sim \mathcal{N}(0, Q_k), v_k \sim \mathcal{N}(0, R_k), \mathbb{E}\left[w_k w_l^T\right] = Q_k \delta_{kl}$, and $\mathbb{E}\left[v_k v_l^T\right] = R_k \delta_{kl}$
- R_k is a diagonal matrix. Inverse of R_k is the precision matrix

Augmented model

We consider each of the q time steps as a single time step



$$
\begin{aligned}
\boldsymbol{X}_k &:= [\boldsymbol{x}_{kq-q+1}^T, \ldots, \boldsymbol{x}_{kq}^T]^T, \\
\boldsymbol{Y}_k &:= [\boldsymbol{y}_{kq-q+1}^T, \ldots, \boldsymbol{y}_{kq}^T]^T, \\
\boldsymbol{W}_k &\sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{Q}}_k), \ \boldsymbol{V}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{R}}_k), \\
\boldsymbol{\mathcal{Q}}_k &:= \text{diag}([\boldsymbol{Q}_{kq-q+1}, \ldots, \boldsymbol{Q}_{kq+q-1}]), \\
\boldsymbol{\mathcal{R}}_k &:= \text{diag}([\boldsymbol{R}_{kq-q+1}, \ldots, \boldsymbol{R}_{kq}]),
\end{aligned}
$$

(2a)

(2b)

$$
\boldsymbol{X}_{k+1} = \boldsymbol{F}_k(\boldsymbol{X}_k, \boldsymbol{W}_k), \ \boldsymbol{Y}_k = \boldsymbol{H}_k(\boldsymbol{X}_k) + \boldsymbol{V}_k, \tag{3}
$$

The filtering process consists of two sequential steps: a) time update and b) measurement update.

EnKF— random samples generated using Monte Carlo techniques[1]

UKF — minimal set of deterministic samples along with their weights[2]

Covariance Update:

$$\boldsymbol{\Sigma}^{+}_{xx,k+1} = \boldsymbol{\Sigma}^{-}_{xx,k+1} - \mathcal{K}\boldsymbol{\Sigma}^{-T}_{xy,k+1}. \qquad (4)$$
$$\mathcal{K} := \boldsymbol{\Sigma}^{-}_{xy,k+1}(\boldsymbol{\Sigma}^{-}_{yy,k+1} + \mathcal{R}_k)^{-1}$$

[1] Geir Evensen and Peter Jan Van Leeuwen. "Assimilation of Geosat altimeter data for the Agulhas current using the ensemble Kalman filter with a quasigeostrophic model". In: *Monthly Weather Review* 124.1 (1996), pp. 85–96; Youmin Tang, Jaison Ambandan, and Dake Chen. "Nonlinear measurement function in the ensemble Kalman filter". In: *Advances in Atmospheric Sciences* 31.3 (2014), pp. 551–558. DOI: 10.1007/s00376-013-3117-9.

[2] Eric A Wan and Rudolph Van Der Merwe. "The unscented Kalman filter for nonlinear estimation". In: *Proceedings of the IEEE 2000 Adaptive Systems for Signal Processing, Communications, and Control Symposium (Cat. No. 00EX373)*. Ieee. 2000, pp. 153–158.

Covariance update equation (4) for time step k, can be written as:

$$\begin{aligned}\boldsymbol{\Sigma}^{+}_{xx,k} &= \boldsymbol{\Sigma}^{-}_{xx,k} - \boldsymbol{\Sigma}^{-}_{xy,k}(\boldsymbol{\Sigma}^{-}_{yy,k} + \boldsymbol{\mathcal{S}}_k^{-1})^{-1}\boldsymbol{\Sigma}^{-T}_{xy,k} \\ &= \boldsymbol{\Sigma}^{-}_{xx,k} - \boldsymbol{\Sigma}^{-}_{xy,k}\left(\boldsymbol{\Sigma}^{-}_{yy,k} + \text{diag}([\lambda_1, \dots, \lambda_{qn_y}])^{-1}\right)^{-1}\boldsymbol{\Sigma}^{-T}_{xy,k}\end{aligned}$$

- Sensor precision of i^{th} sensor is λ_i for the augmented system, which are control variables regulating $\boldsymbol{\Sigma}^{+}_{xx,k}$.
- Objective is to design $\{\lambda_i\}$ such that $\boldsymbol{M}_q\boldsymbol{\Sigma}^{+}_{xx,k}\boldsymbol{M}_q^T \preceq \boldsymbol{P}^d_{kq}$
- $\boldsymbol{M}_q := [\mathbf{0}^1_{n\times n}, \mathbf{0}^2_{n\times n}, \dots, \mathbf{0}^{q-1}_{n\times n}, \boldsymbol{I}_{n\times n}]$, is utilized to extract error covariance matrix of posterior estimate of $\boldsymbol{x}_{kq}$ from $\boldsymbol{\Sigma}^{+}_{xx,k}$.

Although we use the augmented model in (3), the performance bound is on the covariance of the estimate of $\boldsymbol{x}_{kq}$.

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Theorem

The optimal precision of each of the sensors, $\boldsymbol{\lambda}_k := [\lambda_1, ..., \lambda_{qn_y}]$ *at time* k*, which guarantees* $\boldsymbol{M}_q \boldsymbol{\Sigma}^+_{xx,k} \boldsymbol{M}_q^T \preceq \boldsymbol{P}^d_{kq}$*, for given posterior ensemble* $\boldsymbol{\mathcal{X}}^+_{k-1}$*, is obtained by solving the following semidefinite programming (SDP) problem,*

$$\boldsymbol{\lambda}^*_k = \min_{\boldsymbol{\lambda_k} := [\lambda_1, ..., \lambda_{qn_y}]^T} ||\boldsymbol{\lambda}_k||_1, \tag{5}$$

subject to,

$$\begin{bmatrix} \boldsymbol{P}^d_{kq} + \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{D} \end{bmatrix} \succeq 0, \quad \lambda_i \geq 0, \ \forall i \in [1, ..., qn_y], \tag{6}$$

where

$$\boldsymbol{A} := -\boldsymbol{M}_q \boldsymbol{\Sigma}^-_{xx,k} \boldsymbol{M}_q^T + \boldsymbol{M}_q \boldsymbol{\Sigma}^-{}_{xy,k} \boldsymbol{\mathcal{S}}_k \boldsymbol{\Sigma}^-{}^T_{xy,k} \boldsymbol{M}_q^T, \boldsymbol{B} := \boldsymbol{M}_q \boldsymbol{\Sigma}^-_{xy,k} \boldsymbol{\mathcal{S}}_k$$
$$\boldsymbol{D} := (\boldsymbol{\Sigma}^-{}_{yy,k})^{-1} + \boldsymbol{\mathcal{S}}_k, \ \boldsymbol{\mathcal{S}}_k := \mathit{diag}([\lambda_1, ..., \lambda_{qn_y}]).$$

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Numerical example

We provide simulation results for the sensor precision selection algorithm for

- single time step update ($q = 1$)
- multiple time step update ($q = 3$)

on The Lorenz (1996) model, also including the case where sensor precisions are constrained.

The Lorentz 1996 model model consists of N_x equally spaced variables, x_i for $i = 1, ..., N_x$, which are evolved as:

$$\frac{dx_i}{dt} = (x_{i+1} - x_{i-2})x_{i-1} - x_i + F, \tag{7}$$

with cyclic boundaries: $x_{i+N} = x_i$ and $x_{i-N} = x_i$.
We fix N_x and F at 20 and 8 respectively, which leads to chaotic behavior in the system dynamics[3].

[3] Edward N Lorenz. "Predictability: A problem partly solved". In: *Proc. Seminar on predictability*. Vol. 1. 1996; Edward N Lorenz and Kerry A Emanuel. "Optimal sites for supplementary weather observations: Simulation with a small model". In: *Journal of the Atmospheric Sciences* 55.3 (1998), pp. 399–414.

Experimental Setup

- We consider $\mathcal{Q}_k = 0$, but with initial condition uncertainty.
- We use $2qN_x + 1$ number of samples for both EnKF and UKF, for $q = 1$ and 3.
- We linearly vary the required error covariance bound from a factor of 0.9 to 0.6 of the initial covariance .

We assume the following non-linear measurement model:

$$y_{i,k} = \frac{1}{1 + e^{-x_{i,k}}} + v_{i,k} \tag{8}$$

where $(.)_{i,k}$ denotes i^{th} component of a vector at time point k, with measurement noise $\boldsymbol{v}_k \sim \mathcal{N}(0, \boldsymbol{R}_k)$.

For $q = 1$ shown in fig.(1) and fig.(2), 21 linearly varying bounds are considered within the interval of $[0.9, 0.6]$.

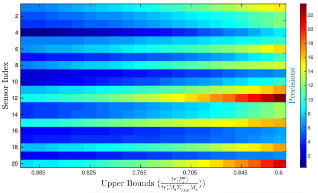

Figure 1: Precision of sensors updated at each time step $(q = 1)$ for EnKF without precision bounds

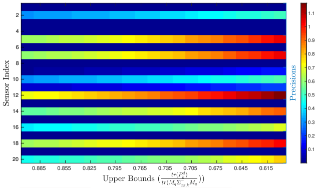

Figure 2: Precision of sensors updated at each time step $(q = 1)$ for UKF without precision bounds

For $q = 3$ shown in fig.(3), fig.(4), 7 linearly varying bounds are chosen from the same interval.

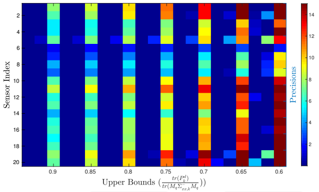

Figure 3: Precision of sensors updated for 3 consecutive time step ($q = 3$), with precision bounds for EnKF

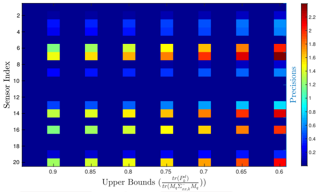

Figure 4: Precision of sensors updated for 3 consecutive time step ($q = 3$), with precision bounds for UKF

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- We use CVX[4] with SeDuMi solver[5] to solve our SDP problem.
- The l_1 norm minimization problem with LMI constraint yields a sparse sensor set.
- We see that the optimal solution results in high accuracy sensing only at the end of the time interval, with poor (or no) sensing within the interval. However, this changes when upper limit on the precisions are reduced. In that case, we will see higher precision within the interval.

[4] Michael Grant and Stephen Boyd. *CVX: Matlab Software for Disciplined Convex Programming, version 2.1*. http://cvxr.com/cvx. Mar. 2014.

[5] Jos F. Sturm. "Using SeDuMi 1.02, A Matlab toolbox for optimization over symmetric cones". In: *Optimization Methods and Software* 11.1-4 (1999), pp. 625–653. DOI: 10.1080/10556789908805766. eprint: https://doi.org/10.1080/10556789908805766.

Conclusion

- A new sensor precision selection algorithm for non-linear dynamical systems is presented in the framework of EnKF and UKF.
- The problem is shown to be convex, which can be easily solved using standard software such as CVX.
- The algorithm is applied to the Lorenz 1996 model of order 20 and results from both EnKF and UKF framework are presented.

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Thank You