Eigen Value Analysis in Lower Bounding Uncertainty of Kalman Filter Estimates

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If the system dynamics is: $\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{w}_k,\ \forall k \in \mathbb{N}$, and the measurement equation is: $\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{n}_k,\ \forall k \in \mathbb{N}$, the Kalman filtering based covariance update equation is:

$$\boldsymbol{P}_{k|k} = \boldsymbol{P}_{k|k-1} - \boldsymbol{P}_{k|k-1}\boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{P}_{k|k-1}\boldsymbol{C}^T + \boldsymbol{R})^{-1}\boldsymbol{C}\boldsymbol{P}_{k|k-1}$$

where $\boldsymbol{P}_{k|k-1}$ and $\boldsymbol{P}_{k|k}$ denotes the prior and posterior covariance.

The question that we are interested in answering is:

How can we calculate $\boldsymbol{R}$ so that steady-state prior covariance $\boldsymbol{P}_\infty \succeq \boldsymbol{P}_l^f$ (lower-bounded)?

Contributions

- We propose a measurement noise ($\boldsymbol{R}$) manipulation scheme to ensure lower-bound on the estimation accuracy of states.
- We have used mathematical tools from eigen value analysis to calculate $\boldsymbol{R}$ that ensures lower-bound on the steady state estimation error of Kalman filter

System model

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{w}_k, \ \forall k \in \mathbb{N}, \tag{1a}$$

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{n}_k, \ \forall k \in \mathbb{N}, \tag{1b}$$

- $\boldsymbol{x}_k \in \mathbb{R}^{n_x}$, $\boldsymbol{y}_k \in \mathbb{R}^{n_y}$, $\boldsymbol{A} \in \mathbb{R}^{n_x \times n_x}$, $\boldsymbol{B} \in \mathbb{R}^{n_x \times n_w}$, $\boldsymbol{C} \in \mathbb{R}^{n_y \times n_x}$.
- The process noise $\boldsymbol{w}_k \in \mathbb{R}^{n_w}$ and measurement noise $\boldsymbol{n}_k \in \mathbb{R}^{n_n}$,is zero-mean Gaussian additive noise with $\mathbb{E}[\boldsymbol{w}_k\boldsymbol{w}_l^T] = \boldsymbol{Q}\delta_{kl}$ and $\mathbb{E}[\boldsymbol{n}_k\boldsymbol{n}_l^T] = \boldsymbol{R}\delta_{kl}$

Unified Algebraic Riccati equation[1] :

$$\begin{aligned} &\boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} + \Delta\boldsymbol{A}^T\boldsymbol{PA} - (\Delta\boldsymbol{A}^T + \boldsymbol{I})\boldsymbol{PB} \\ &\times (\boldsymbol{I} + \Delta\boldsymbol{B}^T\boldsymbol{PB})^{-1}\boldsymbol{B}^T\boldsymbol{P}(\Delta\boldsymbol{A} + \boldsymbol{I}) + \boldsymbol{Q} = 0, \end{aligned} \tag{2}$$

We introduce an extra parameter $\boldsymbol{R} \in \mathbb{R}^{n_y \times n_y}$ in UARE and call it UARE-R. This UARE-R:

$$\begin{aligned} &\boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} + \Delta\boldsymbol{A}^T\boldsymbol{PA} - (\Delta\boldsymbol{A}^T + \boldsymbol{I})\boldsymbol{PB} \\ &\times (\boldsymbol{R} + \Delta\boldsymbol{B}^T\boldsymbol{PB})^{-1}\boldsymbol{B}^T\boldsymbol{P}(\Delta\boldsymbol{A} + \boldsymbol{I}) + \boldsymbol{Q} = 0, \end{aligned} \tag{3}$$

is often encountered in optimal control[2] and estimation problems[3].

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[1] Richard H Middleton and Graham C Goodwin. *Digital Control and Estimation: A Unified Approach (Prentice Hall Information and System Sciences Series)*. Prentice Hall Englewood Cliffs, NJ, 1990.

[2] Arthur Earl Bryson. *Applied optimal control: optimization, estimation and control*. Routledge, 2018.

[3] Brian DO Anderson and John B Moore. "Optimal filtering". In: *Englewood Cliffs* 21 (1979), pp. 22–95.

In UARE-R

- Using $\Delta = 0$, replacing $\boldsymbol{A}$ by $\boldsymbol{A}^T$, and $\boldsymbol{B}$ by $\boldsymbol{C}^T$, we recover the Continuous Time Algebraic Riccati equation (CARE), solution to which gives us the steady state covariance for a Kalman-Bucy filter.
- Using $\Delta = 1$, replacing $\boldsymbol{A} + \boldsymbol{I}$ by $\boldsymbol{A}^T$, and $\boldsymbol{B}$ by $\boldsymbol{C}^T$ we recover the Discrete Algebraic Riccati equation (DARE) associated with steady state covariance of the Kalman Filter, where $\boldsymbol{P}$ denotes the steady-state error covariance matrix.

$$\boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^T - \boldsymbol{P} - \boldsymbol{A}\boldsymbol{P}\boldsymbol{C}^T(\boldsymbol{R} + \boldsymbol{C}\boldsymbol{P}\boldsymbol{C}^T)^{-1}\boldsymbol{C}\boldsymbol{P}\boldsymbol{A}^T + \boldsymbol{Q} = 0,$$

Theorem

Let $\boldsymbol{P}$ be the positive solution of the UARE-R (3), then

$$\boldsymbol{P} \succeq (\Delta\boldsymbol{A} + \boldsymbol{I})^T (\boldsymbol{P}_{l0}^{-1} + \Delta\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T)^{-1} (\Delta\boldsymbol{A} + \boldsymbol{I}) + \Delta\boldsymbol{Q} \equiv \boldsymbol{P}_{l1} \quad (4)$$

where the matrix $\boldsymbol{P}_{l0}$ is defined as,

$$\boldsymbol{P}_{l0} \equiv (\Delta\boldsymbol{A} + \boldsymbol{I})^T (\varphi^{-1}\boldsymbol{I} + \Delta\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T)^{-1} (\Delta\boldsymbol{A} + \boldsymbol{I}) + \Delta\boldsymbol{Q} \quad (5)$$

and the positive constant φ is defined as,

$$\begin{aligned}\varphi \equiv & f(-[\lambda_{n_x}(\boldsymbol{A} + \boldsymbol{A}^T + \Delta\boldsymbol{A}^T\boldsymbol{A}) + \Delta\lambda_{n_x}(\boldsymbol{Q})\lambda_1(\boldsymbol{R}^{-1}) \\ & \times \sigma_1^2(\boldsymbol{B})], 2\lambda_1(\boldsymbol{R}^{-1})\sigma_1^2(\boldsymbol{B}), 2\lambda_{n_x}(\boldsymbol{Q})),\end{aligned} \quad (6)$$

where $f(a,b,c)$ is defined as,

$$f(a,b,c) \equiv \frac{-a + \sqrt{a^2 + bc}}{b}. \quad (7)$$

Theorem

For a given scalar cost function c($\boldsymbol{R}$) and an lower bound $(1/\lambda_u^f)$ on the spectrum of $\boldsymbol{R}$, the solution $\boldsymbol{R}^*$, whose spectrum is $\lambda(\boldsymbol{R}^*) := \{\lambda_1 \geq \cdots \geq \lambda_{ny}\}$, where $\lambda_{ny} \geq (1/\lambda_u^f)$, that satisfies a given lower bound $\boldsymbol{P}_l^f$ on the steady state prior covariance matrix $\boldsymbol{P}$ of Kalman filter, is given by the following optimization problem.

$$\boldsymbol{R}^* := \operatorname*{argmin}_{\boldsymbol{R}} c(\boldsymbol{R})$$

Such that,

$$\boldsymbol{R} \succeq (1/\lambda_u^f)\boldsymbol{I}, \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{C}^T \\ \boldsymbol{C} & \boldsymbol{R} \end{bmatrix} \succeq 0,$$

where,

$$\boldsymbol{T}_1 \equiv \boldsymbol{A}^T(\boldsymbol{P}_l^f - \boldsymbol{Q})^{-1}\boldsymbol{A} - {\boldsymbol{P}'_{l0}}^{-1}, \boldsymbol{P}'_{l0} \equiv \boldsymbol{A}(\varphi'^{-1}\boldsymbol{I} + \lambda_u^f \boldsymbol{C}^T\boldsymbol{C})^{-1}\boldsymbol{A}^T + \boldsymbol{Q}.$$

$$\varphi' \equiv f(-[\lambda_{n_x}(\boldsymbol{A}\boldsymbol{A}^T - \boldsymbol{I}) + \lambda_{n_x}(\boldsymbol{Q})\lambda_u^f\sigma_1^2(\boldsymbol{C}^T)], 2\lambda_u^f\sigma_1^2(\boldsymbol{C}^T), 2\lambda_{n_x}(\boldsymbol{Q})),$$

The prescribed $\mathbf{P}_l^f$ should lie between $\mathbf{P}^{lb}$ and $\mathbf{P}^{ub}$ satisfying the following:

$$\mathbf{P}^{lb} := \boldsymbol{A}(\mathbf{P}^{lb} - \mathbf{P}^{lb}\boldsymbol{C}^T\left[\boldsymbol{C}\mathbf{P}^{lb}\boldsymbol{C}^T\right]^{-1}\boldsymbol{C}\mathbf{P}^{lb})\boldsymbol{A}^T + \boldsymbol{BQB}^T$$

The matrices $\mathbf{P}^{lb}$ is calculated using $\boldsymbol{R} = \mathbf{0}$ in the DARE. When $\boldsymbol{R} = \mathbf{0}$, the DARE is solved using generalized Shur method as in[4] on an extended matrix pencil. The covariance $\mathbf{P}^{ub}$ satisfies the following:

$$\mathbf{P}^{ub} := (\boldsymbol{A}\mathbf{P}^{ub}\boldsymbol{A}^T + \boldsymbol{BQB}^T)$$

The matrix $\mathbf{P}^{ub}$ is calculated by using $\boldsymbol{R} = \infty$ in the DARE. An unique $\mathbf{P}^{ub}$ exists if $\boldsymbol{A}$ is stable.

[4] Vasile Sima and Peter Benner. "Solving linear matrix equations with SLICOT". In: *European Control Conference, ECC 2003* (Apr. 2015).

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{w}_k,$$
$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{n}_k.$$

- Dimension of $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ is both 10.
- The $\boldsymbol{B}$ and $\boldsymbol{Q}$ matrices are chosen to be $\boldsymbol{I}$.
- The $\boldsymbol{C}$ matrices are chosen to be $2\boldsymbol{I}$.
- The $\boldsymbol{A}$ and $\boldsymbol{C}$ matrices are chosen such that $[\boldsymbol{A}, \boldsymbol{C}]$ pair is detectable and $[\boldsymbol{A}, \boldsymbol{B}\boldsymbol{Q}^{1/2}]$ pair is stabilizable.
- $\boldsymbol{R}$ is a diagonal matrix.

- The matrices $\mathbf{P}^{lb}$ and $\mathbf{P}^{ub}$ are first calculated. The eigen values of

  $$\text{eig}(\mathbf{P}^{ub}) = [1.000\ 1.001\ 1.012\ 1.123\ 1.186\ 2.139\ 3.172\ 4.705\ 9.096\ 279.143],$$

  while the eigenvalues of $\mathbf{P}^{lb}$ all are equal to 1.
- Prescribed lower bound is $\boldsymbol{P}_l^f$ to be $(1/16)(\mathbf{P}^{ub} + 15\mathbf{P}^{lb})$.
- We calculate $\varphi' = 1.0000193$ and $\boldsymbol{P}'_{u0}$. We select the upper bound λ_u^f to be 0.03.
- The eigen values of $\boldsymbol{P}'_{l0}$:

  $$\text{eig}(\boldsymbol{P}'_{l0}) = [28.689\ 2.601\ 2.028\ 1.599\ 1.480\ 1.103\ 1.078\ 1.006\ 1.000\ 1.000].$$

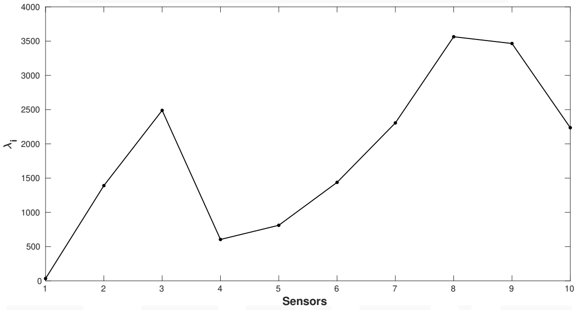

Figure 1: Plot of sensor covariance values for 10 sensors for prescribed lower bound on $\boldsymbol{P}$. Circle denotes covariance values calculated from minimization of l_1 norm of the vector $\boldsymbol{\lambda}$

We solve the optimization problems using CVX in Matlab. The minimum l_1 norm cost is 18336.433 . On a 2GHz Intel Core i5 machine, the l_1 problem takes 1.20 seconds.

Results II

- The solution is verified by calculating the eigen values of the $\boldsymbol{P} - \boldsymbol{P}_l^f$ matrix, which turns out to be all positive, where $\boldsymbol{P}$ is the DARE solution for the optimal $\boldsymbol{R}$
- We notice that there is a large gap between the lower bound and the final steady state value of $\boldsymbol{P}$. This is due to the fact that we used eigen value approximations in deriving the result.
- An ad-hoc method to reduce this gap is to iteratively reduce the magnitude of the $\boldsymbol{\lambda}$ till the eigenvalues of $\boldsymbol{P} - \boldsymbol{P}_l^f$ remain all positive. We found out that we can reduce the $\boldsymbol{\lambda}$ by a factor of 0.08 and still ensure $\boldsymbol{P} \succeq \boldsymbol{P}_l^f$.

Conclusion

- We formulate an methodology to calculate the measurement noise covariance which ensures that the steady state error covariance of the state estimates are lower-bounded by a prescribed bound.
- We introduce a modified Unified Algebraic Riccati Equation (UARE-R) and exploit eigen value analysis to construct a feasible set of measurement noise covariance.

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Thank You